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COMPACT COMPOSITION OPERATORS ON THE BLOCH SPACE

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ABSTRACT. Necessary and sufficient conditions are given for a composition operator $C_\phi f = f \circ \phi$ to be compact on the Bloch space $\mathscr{B}$ and on the little Bloch space $\mathscr{B}_0$. Weakly compact composition operators on $\mathscr{B}_0$ are shown to be compact. If $\phi \in \mathscr{B}_0$ is a conformal mapping of the unit disk $\mathbb{D}$ into itself whose image $\phi(\mathbb{D})$ approaches the unit circle $\mathbb{T}$ only in a finite number of nontangential cusps, then C_ϕ is compact on $\mathscr{B}_0$. On the other hand if there is a point of $\mathbb{T} \cap \overline{\phi(\mathbb{D})}$ at which $\phi(\mathbb{D})$ does not have a cusp, then C_ϕ is not compact.

1. INTRODUCTION

Let $\mathbb{D}$ denote the unit disk in the complex plane. A function f holomorphic in $\mathbb{D}$ is said to belong to the Bloch space $\mathscr{B}$ if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$$

and to the little Bloch space $\mathscr{B}_0$ if

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

It is well known that $\mathscr{B}$ is a Banach space under the norm

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$$

and that $\mathscr{B}_0$ is a closed subspace of $\mathscr{B}$. Furthermore, $\mathscr{B}$ is isometrically isomorphic to the second dual of $\mathscr{B}_0$ and the inclusion $\mathscr{B}_0 \subset \mathscr{B}$ corresponds to the canonical imbedding of $\mathscr{B}_0$ into $\mathscr{B}_0^{**}$ [ACP]. It is a simple consequence of the Schwarz-Pick lemma [A] that a holomorphic mapping ϕ of the unit disk into itself induces a bounded composition operator $C_\phi f = f\circ\phi$ on $\mathscr{B}$. Indeed, if $f \in \mathscr{B}$, then

$$\begin{aligned}(1-|z|^2)|(f\circ\phi)'(z)| &= (1-|z|^2)|f'(\phi(z))||\phi'(z)| \\ &= \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|(1-|\phi(z)|^2)|f'(\phi(z))|\end{aligned} \tag{1}$$

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and the Schwarz-Pick lemma guarantees that

$$\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| \leq 1. \tag{2}$$

Since the identity function $f(z)=z$ belongs to $\mathscr{B}_0$, it is clear that $\phi \in \mathscr{B}_0$ if C_ϕ maps $\mathscr{B}_0$ into itself. Conversely, if $\phi \in \mathscr{B}_0$ and $f \in \mathscr{B}_0$, it follows from (1) and (2) that $f \circ \phi \in \mathscr{B}_0$. Indeed, if $\epsilon > 0$, there exists $\delta > 0$ such that $(1-|z|^2)|f'(z)| < \epsilon$ whenever $|z|^2 > 1-\delta$. In particular, $(1-|z|^2)|(f\circ\phi)'(z)| < \epsilon$ whenever $|\phi(z)|^2 > 1-\delta$. On the other hand, if $|\phi(z)|^2 \leq 1-\delta$,

$$(1-|z|^2)|(f\circ\phi)'(z)| \leq \frac{\|f\|_{\mathscr{B}}}{\delta}(1-|z|^2)|\phi'(z)|,$$

and the right-hand side tends to 0 as $|z| \to 1$.

In Section 2 the compact composition operators on $\mathscr{B}_0$ and on $\mathscr{B}$ will be characterized in terms of the quotient $\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|$. A bounded linear operator $T: X \to Y$ from the Banach space X to the Banach space Y is *weakly compact* if T takes bounded sets in X into relatively weakly compact sets in Y. Gantmacher's theorem [D, p. 21] asserts that T is weakly compact if and only if $T^{**}(X^{**}) \subset Y$ where T^{**} denotes the second adjoint of T. This theorem and the characterization of compact operators on $\mathscr{B}_0$ will be used to show that every weakly compact composition operator on $\mathscr{B}_0$ is compact.

In Section 3 the results of Section 2 will be applied to certain univalent functions ϕ which map $\mathbb{D}$ into itself. It is known that such functions belong to $\mathscr{B}_0$ [P, p. 12]; and it will be clear from Section 2 that if $\|\phi\|_\infty < 1$, then C_ϕ is compact on $\mathscr{B}_0$. On the other hand if $\|\phi\|_\infty = 1$ and there is a point of $\mathbb{T}\cap\overline{\phi(\mathbb{D})}$ at which $\phi(\mathbb{D})$ does not have a cusp, then C_ϕ is not compact. However if $\mathbb{T}\cap\overline{\phi(\mathbb{D})}$ consists of only one point at which $\phi(\mathbb{D})$ has a nontangential cusp, then C_ϕ is compact on $\mathscr{B}_0$.

2. Compactness

Theorem 1 gives a precise description of those ϕ which induce compact composition operators on $\mathscr{B}_0$. It will be useful first to give a criterion for compactness in $\mathscr{B}_0$.

Lemma 1. *A closed set K in $\mathscr{B}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K}(1-|z|^2)|f'(z)| = 0. \tag{3}$$

Proof. First suppose that K is compact and let $\epsilon > 0$. Choose an $\epsilon/2$-net f_1, f_2, ..., f_n in K. There is an r, $0<r<1$, such that $(1-|z|^2)|f_i'(z)| < \epsilon/2$ if $|z|>r$, $1 \leq i \leq n$. If $f \in K$, $\|f-f_i\|_{\mathscr{B}} < \epsilon/2$ for some f_i and so

$$(1-|z|^2)|f'(z)| \leq \|f-f_i\|_{\mathscr{B}} + (1-|z|^2)|f_i'(z)| < \epsilon$$

whenever $|z|>r$. This establishes (3).

On the other hand if K is a closed bounded set which satisfies (3) and (f_n) is a sequence in K, then by Montel's theorem there is a subsequence (f_{n_k}) which converges uniformly on compact subsets of $\mathbb{D}$ to some holomorphic function f. Then also (f'_{n_k}) converges uniformly to f' on compact subsets of $\mathbb{D}$. By (3), if

$\epsilon > 0$, there is an r, $0 < r < 1$, such that for all $g \in K$, $(1-|z|^2)|g'(z)| < \epsilon/2$ if $|z| > r$. It follows that $(1-|z|^2)|f'(z)| < \epsilon/2$ if $|z| > r$. Since (f_{n_k}) converges uniformly to f and (f'_{n_k}) converges uniformly to f' on $|z| \leq r$, it follows that $\limsup_{k\to\infty} \|f_{n_k} - f\|_{\mathscr{B}} \leq \epsilon$. Since $\epsilon > 0$, $\lim_{k\to\infty} \|f_{n_k} - f\|_{\mathscr{B}} = 0$ and so K is compact.

Theorem 1. *If ϕ is a holomorphic mapping of the unit disk $\mathbb{D}$ into itself, then ϕ induces a compact composition operator on $\mathscr{B}_0$ if and only if*

$$\lim_{|z|\to 1} \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| = 0. \tag{4}$$

Proof. It follows from Lemma 1 that C_ϕ is compact on $\mathscr{B}_0$ if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathscr{B}}\leq 1} (1-|z|^2)|(f\circ\phi)'(z)| = 0.$$

But

$$(1-|z|^2)|(f\circ\phi)'(z)| = \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|(1-|\phi(z)|^2)|f'(\phi(z))|,$$

and

$$\sup_{\|f\|_{\mathscr{B}}\leq 1} (1-|w|^2)|f'(w)| = 1$$

for each $w \in \mathbb{D}$. The theorem follows.

It should be remarked that (4) implies $\phi \in \mathscr{B}_0$. A similar condition characterizes compact composition operators on $\mathscr{B}$.

Theorem 2. *If ϕ is a holomorphic mapping of the unit disk $\mathbb{D}$ into itself, then ϕ induces a compact composition operator on $\mathscr{B}$ if and only if for every $\epsilon > 0$, there exists r, $0 < r < 1$, such that*

$$\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| < \epsilon \tag{5}$$

whenever $|\phi(z)| > r$.

Proof. First assume that (5) holds. In order to prove that C_ϕ is compact on $\mathscr{B}$ it is enough to show that if (f_n) is a bounded sequence in $\mathscr{B}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$, then $\|f_n\circ\phi\|_{\mathscr{B}} \to 0$. Let $M = \sup_n \|f_n\|_{\mathscr{B}}$. Given $\epsilon > 0$ there exists r, $0 < r < 1$, such that $\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| < \frac{\epsilon}{2M}$ if $|\phi(z)| > r$. Since

$$\begin{aligned}(1-|z|^2)|(f_n\circ\phi)'(z)| &= \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|(1-|\phi(z)|^2)|f_n'(\phi(z))| \\ &\leq M\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|,\end{aligned}$$

it follows that $(1-|z|^2)|(f_n\circ\phi)'(z)| < \frac{\epsilon}{2}$ if $|\phi(z)| > r$. On the other hand, $f_n\circ\phi(0) \to 0$ and $(1-|w|^2)|f_n'(w)| \to 0$ uniformly for $|w| \leq r$. Since

$$(1-|z|^2)|(f_n\circ\phi)'(z)| \leq (1-|\phi(z)|^2)|f_n'(\phi(z))|,$$

it follows that for large enough n, $|f_n\circ\phi(0)| < \frac{\epsilon}{2}$ and $(1-|z|^2)|(f_n\circ\phi)'(z)| < \frac{\epsilon}{2}$ if $|\phi(z)| \leq r$. Hence $\|f_n\circ\phi\|_{\mathscr{B}} < \epsilon$ for large n.

Now assume that (5) fails. Then there exists a subsequence (z_n) in $\mathbb{D}$ and an $\epsilon > 0$ such that $|z_n| \to 1$ and $\frac{1-|z_n|^2}{1-|\phi(z_n)|^2}|\phi'(z_n)| > \epsilon$ for all n. Passing to a subsequence if necessary it may be assumed that $w_n = \phi(z_n) \to w_0 \in \mathbb{T}$. Let $f_n(z) = \log \frac{1}{1-\overline{w}_n z}$. Then (f_n) converges to f_0 uniformly on compact subsets of $\mathbb{D}$. On the other hand,

$$\begin{aligned}\|C_\phi f_n - C_\phi f_0\|_{\mathscr{B}} &\geq (1-|z_n|^2)|(C_\phi f_n)'(z_n) - (C_\phi f_0)'(z_n)| \\ &= (1-|z_n|^2)|\phi'(z_n)| \left| \frac{\overline{w}_n}{1-|w_n|^2} - \frac{\overline{w}_0}{1-\overline{w}_0 w_n} \right| \\ &= \frac{(1-|z_n|^2)}{1-|w_n|^2}|\phi'(z_n)| \left| \frac{\overline{w}_n - \overline{w}_0}{1-\overline{w}_0 w_n} \right| \\ &> \epsilon\end{aligned}$$

for all n, so $C_\phi f_n$ does not converge to $C_\phi f_0$ in norm. Hence C_ϕ is not compact.

It is important to note that although (4) implies (5), since in this case C_ϕ on $\mathscr{B}$ is the second adjoint of C_ϕ on $\mathscr{B}_0$, the two conditions are not equivalent. Condition (4) implies that $\phi \in \mathscr{B}_0$, while there certainly exist functions $\phi \notin \mathscr{B}_0$ which satisfy (5). Indeed, any ϕ for which $\|\phi\|_\infty < 1$ satisfies (5) trivially.

A sequence (w_n) in $\mathbb{D}$ is said to be η-separated if $\rho(w_n, w_m) = \left|\frac{w_m - w_n}{1-\overline{w}_m w_n}\right| > \eta$ whenever $m \neq n$. Thus an η-separated sequence consists of points which are uniformly far apart in the pseudohyperbolic metric on $\mathbb{D}$, or equivalently, the hyperbolic balls $\Delta(w_n, r) = \{z \mid \rho(z, w_n) < r\}$ are pairwise disjoint for some $r > 0$. Evidently any sequence (w_n) in $\mathbb{D}$ which satisfies $|w_n| \to 1$ possesses an η-separated subsequence for any $\eta > 0$. In particular, if the sequence (w_n) in the proof of Theorem 2 is η-separated, then the calculation in the proof shows that $\|C_\phi f_m - C_\phi f_n\| > \epsilon\eta$ whenever $m \neq n$, so $(C_\phi f_n)$ has no norm convergent subsequences.

Another property of separated sequences is contained in the next proposition. This proposition is related to some interpolation results of Rochberg [RR1, RR2]. Since the method of proof is precisely the same as Rochberg's, a proof will only be sketched.

Proposition 1. *There is an absolute constant $R > 0$ such that if (w_n) is R-separated, then for every bounded sequence (λ_n) there is an $f \in \mathscr{B}$ such that $(1-|w_n|^2)f'(w_n) = \lambda_n$ for all n.*

The idea of the proof is to consider two operators $S: \mathscr{B} \to l^\infty$ given by

$$S(f)_n = (1-|w_n|^2)f'(w_n)$$

and $T: l^\infty \to \mathscr{B}$ given by

$$T(\lambda)(z) = \sum_{n=1}^{\infty} \lambda_n \frac{1}{3\overline{w}_n} \frac{(1-|w_n|^2)^3}{(1-\overline{w}_n z)^3}$$

where $\lambda = (\lambda_n) \in l^\infty$. The proposition will follow if it can be shown that $\|I - ST\| < 1$, for then ST will be invertible and so S will be onto. The symbol C will denote a constant whose value changes from place to place but does not depend on R. Now

$$(ST-I)(\lambda)_n = (1-|w_n|^2)\sum_{m\neq n}\lambda_m \frac{(1-|w_m|^2)^3}{(1-\overline{w}_m w_n)^4}$$

and so it will be enough to estimate

$$\sup_n (1-|w_n|)\sum_{m\neq n}\frac{(1-|w_m|^2)^3}{|1-\overline{w}_m w_n|^4}.$$

If $R > 1/2$, say, then there is a fixed $\delta > 0$ such that the Euclidean disk D_m of center w_m and radius $\delta(1-|w_m|^2)$ is contained in the hyperbolic disk $\Delta_m = \Delta(w_m, R)$ and is disjoint from the hyperbolic disks Δ_n for $n \neq m$. Since $|1-\overline{z}w_n|^{-4}$ is subharmonic and the radius of D_m is comparable to $1-|w_m|^2$,

$$\frac{(1-|w_m|^2)^3}{|1-\overline{w}_m w_n|^4} \leq C\iint_{D_m}\frac{1-|w_m|^2}{|1-\overline{z}w_n|^4}\,dx\,dy;$$

and since $|1-\overline{w}_n z|$ dominates $1-|w_m|^2$ on D_m, it follows that

$$\frac{(1-|w_m|^2)^3}{|1-\overline{w}_m w_n|^4} \leq C\iint_{D_m}\frac{1}{|1-\overline{w}_n z|^3}\,dx\,dy$$

and hence

$$\begin{aligned}\sup_n (1-|w_n|)\sum_{m\neq n}\frac{(1-|w_m|^2)^3}{|1-\overline{w}_m w_n|^4} &\leq C\iint_{\cup_{m\neq n} D_m}\frac{1-|w_n|^2}{|1-\overline{w}_n z|^3}\,dx\,dy \\ &\leq C\iint_{\mathbb{D}\setminus\Delta_n}\frac{1-|w_n|^2}{|1-\overline{w}_n z|^3}\,dx\,dy.\end{aligned}$$

The change of variables $z = \frac{w_n+\zeta}{1+\overline{w}_n\zeta}$ turns this into

$$\sup_n (1-|w_n|)\sum_{m\neq n}\frac{(1-|w_m|^2)^3}{|1-\overline{w}_m w_n|^4} \leq C\iint_{|\zeta|>R}\frac{1}{|1+\overline{w}_n\zeta|}\,d\xi\,d\eta,$$

and the last integral can be made arbitrarily small uniformly in n if R is chosen close enough to 1. This provides the desired estimate.

Since every sequence (w_n) with $|w_n| \to 1$ contains an R-separated subsequence (w_{n_k}), it follows that there is an $f \in \mathscr{B}$ such that $(1-|w_{n_k}|^2)f'(w_{n_k}) = 1$ for all k. This will be used in the proof of the next theorem.

Theorem 3. *Every weakly compact composition operator C_ϕ on $\mathscr{B}_0$ is compact.*

Proof. The composition operator $C_\phi: \mathscr{B}_0 \to \mathscr{B}_0$ is compact if and only if

$$\lim_{|z|\to 1}\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| = 0$$

and, according to Gantmacher's theorem, weakly compact if and only if $C_\phi f \in \mathscr{B}_0$ for every $f \in \mathscr{B}$. If C_ϕ is not compact, there is an $\epsilon > 0$ and a sequence (z_n), $|z_n| \to 1$, such that

$$\frac{1-|z_n|^2}{1-|\phi(z_n)|^2}|\phi'(z_n)| \geq \epsilon$$

for all n. Since $\phi \in \mathscr{B}_0$, $|\phi(z_n)| \to 1$, and by passing to a subsequence it may be assumed that $(\phi(z_n))$ is R-separated. If $f \in \mathscr{B}$,

$$(1-|z_n|^2)|(C_\phi f)'(z_n)| = \frac{1-|z_n|^2}{1-|\phi(z_n)|^2}|\phi'(z_n)|(1-|\phi(z_n)|^2)|f'(\phi(z_n))| \\ \geq \epsilon(1-|\phi(z_n)|^2)|f'(\phi(z_n))|.$$

Since $(\phi(z_n))$ is R-separated, an application of Proposition 1 produces an $f \in \mathscr{B}$ such that $(1-|\phi(z_n)|^2)|(C_\phi f)'(z_n)| = 1$, for all n. Since $(1-|z_n|^2)|(C_\phi f)'(z_n)| \geq \epsilon$ and $|z_n| \to 1$, $C_\phi f \notin \mathscr{B}_0$ and so C_ϕ is not weakly compact.

A slight refinement of these arguments will show that a noncompact composition operator on $\mathscr{B}_0$ must be an isomorphism on a subspace isomorphic to the sequence space c_0. This is not surprising since $\mathscr{B}_0$ is known to be isomorphic to c_0.

3. Examples

As remarked in the introduction any holomorphic mapping ϕ of the unit disk into itself satisfying $\|\phi\|_\infty < 1$ induces a compact composition operator on $\mathscr{B}$ and also on $\mathscr{B}_0$ if $\phi \in \mathscr{B}_0$. On the other hand it is easy to see that if ϕ has a finite angular derivative at some point of $\mathbb{T}$, then C_ϕ cannot be compact. Indeed, ϕ has an angular derivative at $\zeta \in \mathbb{T}$ if the nontangential limit $\omega = f(\zeta) \in \mathbb{T}$ exists and if the quotient $\frac{f(z)-f(\zeta)}{z-\zeta}$ converges to some complex number μ as $z \to \zeta$ nontangentially. It is known that $\mu \neq 0$, and the Julia-Carathéodory lemma shows that $\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|$ converges to $\zeta\overline{\omega}\mu \neq 0$ nontangentially. Applying Theorem 1 or 2 as appropriate shows that C_ϕ is not compact

It turns out, however, that ϕ can push the disk much more sharply into itself and still induce a noncompact composition operator. The easiest way to see this is to consider the functions $\phi_{\lambda,\alpha}(z) = 1-\lambda(1-z)^\alpha$, $0 < \lambda, \alpha < 1$. It is easy to see that $\phi_{\lambda,\alpha} \in \mathscr{B}_0$ and that $\phi_{\lambda,\alpha}$ maps $\mathbb{D}$ onto a region which behaves at 1 like a Stolz angle of opening $\pi\alpha$. If C_ϕ were compact on $\mathscr{B}_0$, composition with $\log\frac{1}{1-z}$ would yield a function in $\mathscr{B}_0$, but an easy calculation shows that this is not so. This leads to the consideration of cusps.

Throughout the remainder of this section ϕ will denote a univalent mapping of the unit disk $\mathbb{D}$ into itself with image $G = \phi(\mathbb{D})$. For simplicity it will be assumed that $\overline{G} \cap \mathbb{T} = \{1\}$. The region G is said to have a cusp at 1 [P, p. 256] if

$$\operatorname{dist}(w, \partial G) = o(|1-w|) \tag{6}$$

as $w \to 1$ in G. Otherwise G does not have a cusp at 1. The cusp is said to be nontangential if G lies inside a Stolz angle near 1, i.e., there exist $r, M > 0$ such that

$$|1-w| \leq M(1-|w|^2) \tag{7}$$

if $|1-w| < r$, $w \in G$. Finally the following geometric property of the conformal mapping ϕ will be needed. If ϕ is a conformal mapping with domain $\mathbb{D}$, then

$$\frac{1}{4}(1-|z|^2)|\phi'(z)| \le \operatorname{dist}(\phi(z), \partial G) \le (1-|z|^2)|\phi'(z)|. \tag{8}$$

This inequality, known as the Koebe distortion theorem, is an elementary consequence of the Schwarz lemma and Koebe's one-quarter theorem [G, p. 13]. It can be used to prove that bounded univalent functions lie in $\mathcal{B}_0$. Indeed, if $\phi \notin \mathcal{B}_0$, there is a $\delta > 0$ and a sequence (z_n) in $\mathbb{D}$ with $|z_n| \to 1$ and $(1-|z_n|)|\phi'(z_n)| > \delta$ for all n. Hence $\operatorname{dist}(\phi(z_n), \partial G) > \delta/4$, so $\phi(z_n)$ has a cluster point in G, contradicting the fact that ϕ is a proper map. Theorem 4 provides a negative result.

Theorem 4. *If ϕ is univalent and $G = \phi(\mathbb{D})$ satisfies $\overline{G} \cap \mathbb{T} = \{1\}$ but does not have a cusp at* 1, *then C_ϕ is not compact on $\mathcal{B}_0$.*

Proof. Since G does not have a cusp at 1, (6) fails. Hence there is a $\delta > 0$ and a sequence (z_n) in $\mathbb{D}$ such that $|z_n| \to 1$, but

$$\operatorname{dist}(\phi(z_n), \partial G) \ge \delta|1-\phi(z_n)|.$$

Hence

$$\delta(1-|\phi(z_n)|^2) \le 2\delta(1-|\phi(z_n)|) \le 2\operatorname{dist}(\phi(z_n), \partial G) \le 2(1-|z_n|^2)|\phi'(z_n)|,$$

so $\frac{1-|z_n|^2}{1-|\phi(z_n)|^2}|\phi'(z_n)| \ge \frac{\delta}{2}$. Since $|z_n| \to 1$, Theorem 1 shows that C_ϕ is not compact.

The next theorem shows how to produce compact composition operators on $\mathcal{B}_0$ from univalent mappings ϕ with $\|\phi\|_\infty = 1$.

Theorem 5. *If ϕ is univalent and if G has a nontangential cusp at* 1 *and touches the unit circle at no other point, then C_ϕ is a compact operator on $\mathcal{B}_0$.*

Proof. As $\phi \in \mathcal{B}_0$, it will be enough to show that

$$\lim_{|z|\to 1} \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| = 0,$$

since the theorem will then follow from Theorem 1. Since G has a nontangential cusp at 1, there exist $r, M > 0$ such that

$$|1-w| \le M(1-|w|^2)$$

if $|1-w| < r$, $w \in G$. Let $\epsilon > 0$. Since G has a cusp at 1, there is a $\delta > 0$ such that

$$\operatorname{dist}(w, \partial G) \le \frac{\epsilon}{4M}|1-w|$$

if $|1-w| < \delta$, $w \in G$. Let $\eta = \min(\delta, r)$. If $|1-\phi(z)| < \eta$, it follows that

$$\begin{aligned} \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| &\le \frac{4\operatorname{dist}(\phi(z), \partial G)}{1-|\phi(z)|^2} \\ &\le \frac{\epsilon}{M}\frac{|1-\phi(z)|}{1-|\phi(z)|^2} \\ &< \epsilon. \end{aligned}$$

On the other hand if $|1-\phi(z)| \geq \eta$, there is a constant $N > 0$ such that $|\phi'(z)| \leq N$ by the smoothness assumption and a $\rho > 0$ such that $1-|\phi(z)|^2 \geq \rho$. In this case

$$\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| \leq \frac{N}{\rho}(1-|z|^2),$$

and this is less than ϵ if $|z|^2 > 1-\frac{\rho\epsilon}{N}$. That completes the proof.

It is possible to describe regions G with tangential cusp such that the Riemann mapping $\phi\colon \mathbb{D} \to G$ admits either possibility. Indeed, suppose that $h(\theta)$ and $k(\theta)$ are positive continuous functions on $[0, \theta_0]$ with $h(\theta) = o(\theta)$ and $k(\theta) = o(\theta)$. Let

$$G = \{ re^{i\theta} \mid 0 < \theta < \theta_0,\ h(\theta) < 1-r < h(\theta)+k(\theta) \}.$$

Then clearly G has a tangential cusp at 1. If $k(\theta) = o(h(\theta))$, then, for $w = re^{i\theta} = \phi(z)$,

$$(1-|z|^2)|\phi'(z)| \leq \operatorname{dist}(w, \partial G) \leq k(\theta)$$

and

$$1-|w|^2 \geq 1-|w| > h(\theta),$$

so $\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| \to 0$ as $|\phi(z)| \to 1$. Since ϕ is univalent, the argument of Theorem 5 shows that C_ϕ is compact. On the other hand if $k(\theta) = 2h(\theta)$ and $w(\theta) = (1-2h(\theta))e^{i\theta} = \phi(z(\theta))$, then evidently $\operatorname{dist}(w(\theta), \partial G) > ch(\theta)$ for some constant c, and since $(1-|z|^2)|\phi'(z)| \geq \operatorname{dist}(\phi(z), \partial G)$, it follows that $\frac{1-|z(\theta)|^2}{1-|w(\theta)|^2}|\phi'(z(\theta))| \geq \frac{c}{4}$, and so C_ϕ is not compact.

4. Conclusion

Although the conditions of Theorems 1 and 2 provide succinct analytic conditions on a function ϕ in order that it induce compact composition operators, it is desirable to have more geometric conditions. For example, it is clear from Section 3 that if ϕ is a conformal mapping which has only a finite number of nontangential cusps on the unit circle $\mathbb{T}$ and no other points of contact, then C_ϕ will be compact on $\mathscr{B}_0$. This raises the question of whether or not there is a $\phi \in \mathscr{B}_0$ such that $\overline{\phi(\mathbb{D})} \cap \mathbb{T}$ is infinite and C_ϕ is compact on $\mathscr{B}_0$. In this regard, it is known that if ϕ has nontangential limit of modulus one on a set of positive measure, then ϕ has an angular derivative at some point and so C_ϕ is not compact [Sh, p. 71]. Further information about compact operators considered from a geometric point of view, especially on H^2, can be found in [Sh] and [SSS].

Finally, if $\phi \in \mathscr{B}_0$ and C_ϕ is compact, then $\log \frac{1}{1-\overline{w}\phi(z)} \in \mathscr{B}_0$ for all $w \in \mathbb{T}$. Is the converse of this true?

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